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The extended supersymmetrization of the multicomponent Kadomtsev–Petviashvilli hierarchy

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Abstract. We describe three different approaches to the extended ($N = 2$) supersymmetrization of the multicomponent KP hierarchy. In the first one we utilize only superfermions while in the second only superbosons and in the third superbosons as well as superfermions. It is shown that many soliton equations can be embedded in the supersymmetry theory by using the first approach even if we do not change these equations in the bosonic limit of the supersymmetry. In the second or third approach we obtain a generalization of the soliton equations in the bosonic limit which remains in the class of the usual commuting functions. As a byproduct of our analysis we prove that for the first procedure the bosonic part of the one-component supersymmetric KP hierarchy coincides with the usual classical two-component KP hierarchy.

1. Introduction

Integrable Hamiltonian systems occupy an important place in diverse branches of theoretical physics as exactly solvable models of fundamental physical phenomena ranging from nonlinear hydrodynamics to string theory [1–3]. The general Kadomtsev–Petviashvilli (KP) system [4, 5] is a $(1 + 1)$-dimensional integrable model containing an infinite number of fields. In the Sato approach [6–8], the KP hierarchy is described by the isospectral deformations of the eigenvalue problem $L\Psi = \lambda\psi$ for the pseudodifferential operator $L = \partial + U_2\partial^{-1} + U_3\partial^{-2}$ which is given by

$$L_{t_n} = [B_n, L] \tag{1}$$

where $n = 2, 3, \dots$ and B_n is the differential part of the microdifferential operator L^n. If we require that L satisfies the additional condition that $L^n = B_n, n \geqslant 2$ then the hierarchy of equations given by (1) are reduced to the hierarchy of $(1 + 1)$-dimensional integrable systems called the n-reduced KP hierarchy. For example, the Korteweg–de Vries equation and the Boussinesq equation belong to the two-reduced and three-reduced KP hierarchies, respectively.

On the other hand, a new type of reduction has recently been proposed in a series of articles, which reduces many $(2+1)$-dimensional integrable systems to $(1+1)$-dimensional integrable systems [9–15]. For example, by assuming that L satisfies the constraints

$$L^n = B_n + q\partial^{-1}r \tag{2}$$

we can obtain the so-called k-constrained KP hierarchy. Interestingly, the one-constrained KP hierarchy coincides with the AKNS hierarchy, and the two-constrained KP hierarchy coincides with the Yajima–Oikawa [16] hierarchy. The k-constrained KP hierarchy was shown to possess Lax pairs, recursion operators and bi-Hamiltonian structures [15].

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However, this classification does not exhaust the known generalizations of the KP hierarchies. In this paper we consider two different generalizations of the KP hierarchies. In the next section we describe the so-called multicomponent KP hierarchy and in the succeeding sections we consider the extended supersymmetrization of the multicomponent KP hierarchies.

The idea of using extended supersymmetry (SUSY) for the generalization of the soliton equations appeared almost in parallel to the usage of SUSY in the quantum field theory [17, 18]. The main idea of SUSY is to treat boson and fermion operators equally. The first results, concerned the construction of classical field theories with fermionic and bosonic fields depending on time and one space variable, can be found in [19–22]. In many cases, the addition of fermion fields does not guarantee that the final theory becomes SUSY invariant. Therefore this method was named the fermionic extension in order to distinguish it from the fully SUSY method.

In order to get a SUSY theory we have to add to a system of k bosonic equations kN fermions and $k(N-1)$ boson fields ($k = 1, 2, \ldots,\ \ N = 1, 2, \ldots$) in such a way that the final theory becomes SUSY invariant. Interestingly enough, it appeared that during the supersymmetrizations, some typical SUSY effects (compared with the classical theory) occurred. We mention a few of them: the non-uniqueness of the roots for the SUSY Lax operator [36, 40], the lack of bosonic reduction to the classical equations [35] and the occurence of non-local conservation laws [49, 50]. These effects rely strongly on the descriptions of the generalized classical systems of equations which we would like to supersymmetrize.

From the soliton point of view we can distinguish two important classes of the supersymmetric equations: the non-extended ($N = 1$) and extended ($N > 1$) cases. Consideration of the extended case may imply new bosonic equations whose properties need further investigation. This may be viewed as a bonus, but this extended case is in no way more fundamental than the non-extended one. Indeed, as we show in this paper, it is possible to construct such an extended superymmetric equation which does not contain any new information in the bosonic sector compared with the original non-supersymmetric equation. We carry out the supersymmetrization of the one-component KP hierarchy in two different ways. We show that despite using superfermions in the first approach for the supersymmetrization of the one-component KP hierarchy, the bosonic sector coincides with the usual classical two-component KP hierarchy. Interestingly, the bosonic part of the SUSY Lax pair of the one-component KP hierarchy is a matrix-valued operator, in contrast to the scalar Lax operator in the classical case. Therefore we can claim that we also supersymmetrized the two-component KP hierarchy without any new information in the bosonic sector. We show that in the second case, where we use superbosons, we extend our system to the new bosonic system.

The paper is organized as follows. In section 2 we describe the multicomponent KP hierarchy. Section 3 contains an introduction to the supersymmetrization of this hierarchy which is developed in the sections that follow. In section 4 we describe the superfermionic approach, while in section 5 we describe the superbosonic approach. We use superfermions as well as superbosons in section 6 in the supersymmetrization of our multicomponent KP hierarchy in order to demonstrate the third (mixed) possibility. Section 7 contains concluding remarks.

All calculations presented in this paper have been obtained by extensive application of the symbolic computation language REDUCE.

2. The multicomponent KP hierarchy

The multicomponent KP hierarchy was introduced by Sidorenko and Strampp [14], and is a straightforward generalization of the scalar case. This is a hierarchy associated with the following Lax operator:

$$L_n = \partial^n + u_{n-2}\partial^{n-2} + \cdots + u_0 + \sum_{u=1}^{m} q_i \partial^{-1} r_i \tag{3}$$

The corresponding flows can be constructed by means of the fractional power method [5]. For $n = 1$, one has the multicomponent AKNS hierarchy, which includes the coupled nonlinear Schrödinger equation [51] as an example. For $n = 2$ and $n = 3$ one has the multicomponent Yajima–Oikawa [16] and Melnikov [52] hierarchies, respectively. We consider first the multicomponent AKNS hierarchy, which is given by

$$L = \partial + \sum_{i=1}^{n} q_i \partial^{-1} r_i \tag{4}$$

where the flows are

$$L_{t_k} = \left[\left(L^k\right)_+ , L \right] . \tag{5}$$

The bi-Hamiltonian structure of these equations has been widely discussed in the literature recently [14, 51, 52] and it has the following representation:

$$q_{t_k} = B^0 \frac{\delta H_{k+1}}{\delta q} = B^1 \frac{\delta H_k}{\delta q} \tag{6}$$

where $q = (q_1, q_2, \ldots, q_n, r_1, r_2, \ldots, r_n)$ and

$$B^0 = \begin{pmatrix} O & I \\ -I & O \end{pmatrix} \tag{7}$$

where I is the $m \times m$ identity matrix. B^1 has [54] the form

$$B^1 = \begin{pmatrix} B^1_{11} & B^1_{12} \\ B^1_{21} & B^1_{22} \end{pmatrix} \tag{8}$$

where the $B^1_{n,k} (n, k = 1, 2)$ are $m \times m$ matrices with the elements

$$B^1_{11} = \left\{ q_i \partial^{-1} q_j + q_j \partial^{-1} q_i \right\} \qquad B^1_{12} = \left\{ \left(\partial - \sum_{s=1}^{m} q_s \partial^{-1} r_s \right) \delta_{ij} - q_i \partial^{-1} r_j \right\} \tag{9}$$

$$(B^1_{12})^* = -B^1_{21} \qquad B^1_{22} = \left\{ r_i \partial^{-1} r_j + r_j \partial^{-1} r_i \right\} \tag{10}$$

and $*$ denotes Hermitian conjugation. In the special case $m = 1$ we obtain

$$B^1 = \begin{pmatrix} 2q\partial^{-1}q & \partial - 2q\partial^{-1}r \\ \partial - 2r\partial^{-1}q & 2r\partial^{-1}r \end{pmatrix} \tag{11}$$

Interestingly, this Hamiltonian operator can be considered as the outcome of the Dirac reduction of the Hamiltonian operator connected with the $SL(2, C)$ Kac–Moody algebra [39].

The Hamiltonians H_k may be computed from

$$H_k = \frac{1}{k} \operatorname{Res} \left(L^k\right) \tag{12}$$

where Res denotes the coefficient standing in the ∂^{-1} term.

For the subsequent discussion let us explicitly present equations (6) for the two-component KP hierarchy in two particular cases.

For $k = 2$ these equations are in the form

$$q_{i_t} = q_{ixx} + 2q_i \sum_{i=1}^{m} q_s r_s \tag{13}$$

$$r_{i_t} = -r_{ixx} - 2r_i \sum_{i=1}^{m} q_s r_s \,. \tag{14}$$

This is a vector generalization of the nonlinear Schrödinger equation first considered in [49].

For $k = 3$

$$q_{i_t} = q_{ixxx} + 3q_i \sum_{s=1}^{m} q_{sx} r_s + 3q_{ix} \sum_{s=1}^{m} q_s r_s \tag{15}$$

$$r_{i_t} = r_{ixxx} + 3r_i \sum_{s=1}^{m} q_s r_{sx} + 3r_{ix} \sum_{s=1}^{m} q_s r_s \,. \tag{16}$$

These equations can be further restricted to the known soliton equation. Indeed, assuming that $m = 1$ we obtain the result that equations (12), (13) reduce to the usual nonlinear Schrödinger equation, while equations (14), (15) for $q = r$ reduce to the modified Korteweg–de Vries equation or for $r = 1$ to the Korteweg–de Vries equation.

3. The extended supersymmetrization of the multicomponent KP hierarchy

The basic objects in the supersymmetric analysis are the superfield and the supersymmetric derivative. We will deal with the so-called extended $N = 2$ supersymmetry for which the superfields are superfermions or superbosons that, in addition to x and t, depend upon two anticommuting variables, θ_1 and θ_2, ($\theta_2\theta_1 = -\theta_1\theta_2, \theta_1^2 = \theta_2^2 = 0$). Their Taylor expansion with respect to the θ's is

$$\phi(x, \theta_1, \theta_2) = w(x) + \theta_1\zeta_1(x) + \theta_2\zeta_2(x) + \theta_2\theta_1 u(x) \tag{17}$$

where the fields w, u, are to be interpreted as the boson (fermion) fields for the superboson (superfermion) field ζ_1, ζ_2, and as the fermions (bosons) for the superboson (superfermion) respectively. The superderivatives are defined as

$$\mathcal{D}_1 = \partial_{\theta_1} + \theta_1\partial \qquad \mathcal{D}_2 = \partial_{\theta_2} + \theta_2\partial \tag{18}$$

with the properties

$$\mathcal{D}_2\mathcal{D}_1 + \mathcal{D}_1\mathcal{D}_2 = 0 \qquad \mathcal{D}_1^2 = \mathcal{D}_2^2 = 0. \tag{19}$$

Below we shall use the following notation: $(\mathcal{D}_i F)$ denotes the outcome of the action of the superderivative on the superfield F, while $\mathcal{D}_i F$ denotes the action itself of the superderivative on the superfield F.

The principal problem in the supersymmetrization of soliton equations can be formulated as follows: if we know the evolution equation for the classical function u and its (bi-)Hamiltonian structure or its Lax pair, how can we obtain the evolution equation on the supermultiplet Φ which contains the classical function u? This problem has its own history, and at the moment we have no unique solution. We can distinguish three different methods for its supersymmetrization, namely the algebraic, geometric and direct methods.

In the first two cases we are looking for the symmetry group of the given equation and then we replace this group by the corresponding SUSY group. As a final product we are able to obtain the SUSY generalization of the given equation. Their classification into the algebraic or geometric approach is connected with the kind of symmetry that appears at the classical level. For example, if our classical equation can be described in terms of the geometrical object, then the simple exchange of the classical symmetry group of this object onto the SUSY partner justifies the term geometric. In the algebraic case, we are looking for the symmetry group of this equation without any reference to its geometrical origin. This strategy can be applied to the so-called hidden symmetry, as in the case of the Toda lattice [56], for example.

These methods have both advantages and disadvantages. For example, we sometimes obtain only the fermionic extensions of the given equations [44, 56]. In the case of the extended supersymmetric Korteweg–de Vries equation we have three different fully SUSY extensions; however, only one of them fits these two classifications [26–28, 31].

It seems that the most difficult problem in these approaches is the explanation of why *a priori* a SUSY extension of the classical system of equations should be connected with the SUSY extension of the classical symmetry of these equations. For these reasons we prefer to use the direct approach in which we simply replace all objects which appear in the evolution equation by all possible supermultiplets and superderivatives in such a way as to conserve the gradations of the equation. This is a highly non-unique procedure, and we obtain many different possibilities. However, this arbitrariness can be restricted if in addition we investigate its super-bi-Hamiltonian structure or try to find its supersymmetric Lax pair. This method has been successful in many cases [34, 35, 40–48]. We utilize this method in what follows.

Let us now start trying to find the Lax operator for the multicomponent SUSY KP hierarchy. The direct method suggests that we assume that L depends on the vector supermultiplets F, G, its supersymmetric derivatives and on the derivative and superderivatives in such a way that finally it has the gradation 1. Therefore we postulate that the Lax pair is an operator in the form

$$L = L\,(\partial, D_1, D_2, F, G)\,. \tag{20}$$

In order to specify this form we have to assume the gradations of the supermultiplets F and G. However, we quickly recognize that we encounter three different possibilities of the gradations of F, G:

(i) All F, G are superfermions with the gradation 1/2.
(ii) All F, G are superbosons with the following gradation: F has 0 while G has 1 (or symmetrically).
(iii) A mixture of both previous possibilities: in other words some of the F and G are superbosons and the rest are superfermions.

In the following sections we investigate these possibilities in greater detail.

4. The superfermionic approach

We now assume that the components of the vectors supermultiplets F and G are superfermions which can be written down as

$$F_i = \zeta_i^1 + \theta_1 f_i^1 + \theta_2 f_i^2 + \theta_2\theta_1\zeta_i^2 \tag{21}$$

$$G_i = \eta_i^1 + \theta_1 g_i^1 + \theta_2 g_i^2 + \theta_2\theta_1\eta_i^2 \tag{22}$$

where f_j^i, g_i^k are the usual classical functions while ζ_j^k, η_i^k are Grassmann-valued functions. We choose the Lax operator in such a way that it contains all possible combinations of 'variables' in (20), in such a manner that each term has gradation 1. Then using the symbolic language REDUCE we verified that the following operator:

$$L = \partial + \sum_{i=1}^{k} F_i \cdot \partial^{-1} \cdot D_1 \cdot D_2 \cdot G_i \tag{23}$$

generates the extended supersymmetric multicomponent KP hierarchy. Indeed, its second flow is

$$F_{i_t} = F_{ixx} + 2\sum_{s=1}^{k} F_s\,(\mathcal{D}_1\mathcal{D}_2 G_s F_i) - F_i\left(\sum_{s=1}^{k} F_s G_s\right)^2 \tag{24}$$

$$G_{i_t} = G_{ixx} + 2\sum_{s=1}^{k} G_s\,(\mathcal{D}_1\mathcal{D}_2 G_i F_s) + G_i\left(\sum_{s=1}^{k} F_s G_s\right)^2 \tag{25}$$

while the third is

$$\begin{aligned} F_{i_t} = F_{ixxx} + 3\sum_{j=1}^{k}\Bigg\{ &\left(\mathcal{D}_1\mathcal{D}_2 G_j F_{ix}\right) + \left(\mathcal{D}_1\mathcal{D}_2 G_j F_i\right) F_{jx} \\ &- \sum_{l=1}^{k}\left[\left(\mathcal{D}_1 G_j F_l\right)(\mathcal{D}_1 G_l F_i)\, F_j + \left(\mathcal{D}_2 G_j F_l\right)(\mathcal{D}_2 G_l F_i)\, F_j\right]\Bigg\} - 3F_{ix} Z \end{aligned} \tag{26}$$

$$\begin{aligned} G_{i_t} = G_{ix} + 3\sum_{j=1}^{k}\Bigg\{ &\left(\mathcal{D}_1\mathcal{D}_2 G_{ix} F_j\right) + G_j\left(\mathcal{D}_1\mathcal{D}_2 G_i F_j\right) G_{jx} \\ &+ \sum_{l=1}^{k}\left(\mathcal{D}_1 G_i F_j\right)(\mathcal{D}_1 G_i F_l)\, G_l + (\mathcal{D}_2 G_i)\left(\mathcal{D}_2 G_j F_l\right) G_l\Bigg\} - 3G_{ix} Z \end{aligned} \tag{27}$$

where

$$Z = \sum_{i,j=1}^{k} F_i G_i F_j G_j. \tag{28}$$

Let us now discuss several particular cases of equations (24)–(27). For $k = 1$, equations (24), (25) reduce to

$$F_t = F_{xx} + 2F\,(\mathcal{D}_1\mathcal{D}_2 G F) \tag{29}$$

$$G_t = -G_{xx} - 2G\,(\mathcal{D}_1\mathcal{D}_2 G F)\,. \tag{30}$$

In the components, using (21), (22), we obtain the result that equations (29), (30) are equivalent to

$$\zeta_t^1 = \zeta_{xx}^1 + 2\zeta^1\left(\eta^1\zeta^2 + f^1 g^2 - f^2 g^1\right) \tag{31}$$

$$f_t^1 = f_{xx}^1 - 2\zeta^1\left(g^2\zeta^1 - f^2\eta^2\right)_x + 2f^1\left(\eta^1\zeta^2 + \eta^2\zeta^1 + f^1 g^2 - f^2 g^1\right) \tag{32}$$

$$f_t^2 = f_{xx}^2 + 2\zeta^1\left(g^1\zeta^1 - f^1\eta^1\right)_x + 2f^2\left(\eta^1\zeta^2 + \eta^2\zeta^1 + f^1 g^2 - f^2 g^1\right) \tag{33}$$

$$\zeta_t^2 = \zeta_{xx}^2 - 2\zeta^1 \left(\eta^1\zeta^1\right)_{xx} + 2f^1 \left(g^1\zeta^1 - f^1\eta^1\right)_x$$
$$+2f^2 \left(g^2\zeta^1 - f^2\eta^1\right)_x + 2\zeta^2 \left(\eta^2\zeta^1 + f^1g^2 - f^2g^1\right) \tag{34}$$

$$\eta_t^1 = -\eta_{xx}^1 - 2\eta^1 \left(\eta^2\zeta^1 + f^1g^2 - f^2g^1\right) \tag{35}$$

$$g_t^1 = -g_{xx}^1 + 2\eta^1 \left(g^2\zeta^1 - f^2\eta^1\right)_x - 2g^1 \left(\eta^1\zeta^2 + \eta^2\zeta^1 + f^1g^2 - f^2g^1\right) \tag{36}$$

$$g_t^2 = -g_{xx}^2 - 2\eta^1 \left(g^1\zeta^1 - f^1\eta^1\right)_x - 2g^2 \left(\eta^1\zeta^2 + \eta^2\zeta^1 + f^1g^2 - f^2g^1\right) \tag{37}$$

$$\eta_t^2 = -\eta_{xx}^2 + 2\eta^1 \left(\eta^1\zeta^1\right)_{xx} - 2g^1 \left(g^1\zeta^1 - f^1\eta^1\right)_x$$
$$-2g^2 \left(g^2\zeta^1 - f^2\eta^1\right)_x + 2\eta^2 \left(\eta^1\zeta^2 + f^1g^2 - f^2g^1\right). \tag{38}$$

As we can see, this system of equations can be interpreted as the extended supersymmetric nonlinear Schrödinger equation, which has been extensively discussed recently [36–41, 48]. The bosonic part (in which all fermion fields vanish) gives us equations (7) for $m = 2$ with the following identifications:

$$f^1 = g_1 \qquad f^2 = g_2 \qquad q^1 = -r_2 \qquad q^2 = r_1. \tag{39}$$

Interestingly, our Lax operator in the bosonic limit for $k = 1$ does not reduce to the scalar Lax pair (4). In our case, it has a matrix form

$$L = \begin{pmatrix} \partial + q_1\partial^{-1}r_1 & q_1\partial^{-1}r_2 \\ q_2\partial^{-1}r_1 & \partial + q_2\partial^{-1}r_2 \end{pmatrix}. \tag{40}$$

In this way, we have shown that our one-component extended supersymmetric KP hierarchy in the bosonic sector is equivalent to the usual two-component KP hierarchy. Moreover, in this bosonic sector, our equations constitute the bi-Hamiltonian structure given by (6)–(11), but we are not able to find its supersymmetric bi-Hamiltonian counterparts. As shown in section 3, it was possible using the Dirac reduction technique with the $SL(2, C)$ Kac–Moody algebra to obtain the Hamiltonian operator for the AKNS hierarchy. In the supersymmetric case the situation is more complicated. Indeed it was shown in [58] that it is possible to construct the chiral version of the $N = 2$ SUSY $SL(2, C)$ Kac–Moody algebra. However, this chiral version cannot be applied to our framework, because we do not use chiral fields. Moreover, the application of the direct method to the supersymmetrization of equation (11) or to the $SL(2, C)$ Kac–Moody algebra does not give us the correct solution, a fact we have checked using the symbolic computation program REDUCE (see also [30, 40, 48]).

On the other hand, our equations are Hamiltonian equations which can be written as

$$\begin{pmatrix} F \\ G \end{pmatrix}_{t_k} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \begin{pmatrix} \dfrac{\delta H_k}{\delta F} \\ \dfrac{\delta H_k}{\delta G} \end{pmatrix} \tag{41}$$

where $F = (F_1, F_2, \ldots, F_k)^t$, $G = (G_1, G_2, \ldots, G_k)^t$ and I is a $k \times k$ identity matrix. The Hamiltonians H_k can be computed by using equation (12) in which Res now denotes the coefficient standing in the $\partial^{-1}D_1D_2$ term.

Let us now discuss equations (26), (27) for $k = 1$. In this case they reduce to

$$F_t = F_{xxx} + 3\left[(\mathcal{D}_1\mathcal{D}_2GF_x)\, F + (\mathcal{D}_1\mathcal{D}_2GF)\, F_x\right] \tag{42}$$

$$G_t = G_{xxx} + 3\left[(\mathcal{D}_1\mathcal{D}_2G_xF)\, G + (\mathcal{D}_1\mathcal{D}_2GF)\, G_x\right] \tag{43}$$

with the following bosonic sector:

$$f_{1t} = f_{1xxx} - 3g_1 (f_2 f_1)_x + 3g_2 \left(f_1^2\right)_x \tag{44}$$

$$f_{2t} = f_{2xxx} + 3g_2 (f_1 f_2)_x - 3g_1 \left(f_2^2\right)_x \tag{45}$$

$$g_{1t} = g_{1xxx} + 3f_1 (g_1 g_2)_x - 3f_2 \left(g_1^2\right)_x \tag{46}$$

$$g_{2t} = g_{2xxx} - 3f_2 (g_1 g_2)_x + 3f_1 \left(g_2^2\right)_x . \tag{47}$$

This system of equations can be considered as the vector generalization of the modified Korteweg–de Vries equation. Now we can investigate different reductions of equations (44)–(47) to much simpler equations. For example, by assuming that

$$g_1 = f_1 = f_2 \qquad g_2 = 0 \tag{48}$$

we obtain the usual modified Korteweg–de Vries equation.

To finish this section let us note that the superfermionic method discussed in this section allows us to obtain some extension of the usual system of equations by incorporating anticommuting functions, but we do not change the usual multicomponent KP hierarchy. We show in the next sections that superbosonic or mixed ways of supersymmetrizations generalize our usual multicomponent KP hierarchy in the class of the usual commuting functions.

5. The superbosonic approach

We now assume that the components of the vector supermultiplets F and G are superbosons and can be expressed as

$$F_i = f_i^1 + \theta_1 \zeta_i^1 + \theta_2 \zeta_i^2 + \theta_2 \theta_1 f_i^2 \tag{49}$$

$$G_i = g_i^1 + \theta_1 \eta_i^1 + \theta_2 \eta_i^2 + \theta_2 \theta_1 g_i^2 \tag{50}$$

where ζ_i^j and η_i^j are Grassmann-valued functions while f_i^j, g_i^j are the usual commuting functions. In order to find the proper Lax operator in this case we assume the following gradation on the functions

$$\begin{aligned} &\deg\left(f_i^1\right) = 0 \qquad \deg\left(\zeta_i^j\right) = 0.5 \qquad \deg\left(f_i^2\right) = 1 \\ &\deg\left(g_i^1\right) = 1 \qquad \deg\left(\eta_i^j\right) = 1.5 \qquad \deg\left(g_i^2\right) = 2 . \end{aligned} \tag{51}$$

Note that it is also possible to assume symmetrical gradation in which we replace $f \to g$, $\zeta \to \eta$, but we will not consider such a possibility because we obtain the same information as in the case considered.

We postulate the Lax operator exactly as in (22) and, interestingly, in this case we obtain the same flows, where in contrast to (23) F and G are now superbosons. Therefore, they have different expansions in the components. Let us consider more carefully two particular cases ($k = 1$) of these flows. The second flow is

$$\frac{\mathrm{d}}{\mathrm{d}t} F = F_{xx} - G^2 F^3 + 2F\left(\mathcal{D}_1 \mathcal{D}_2 G F\right) \tag{52}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} G = -G_{xx} + G^3 F^2 - 2G\left(\mathcal{D}_1 \mathcal{D}_2 G F\right) . \tag{53}$$

This is the extended supersymmetric nonlinear Schrödinger equation considered in [48]. The third flow is

$$\frac{\mathrm{d}}{\mathrm{d}t}F = F_{xxx} + 3F_x\,(\mathcal{D}_1\mathcal{D}_2 GF) + 3F\,(\mathcal{D}_1\mathcal{D}_2 GF_x) - 3F^2G^2F_x \tag{54}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}G = G_{xxx} + 3G_x\,(\mathcal{D}_1\mathcal{D}_2 GF) + 3G\,(\mathcal{D}_1\mathcal{D}_2 G_xF) - 3F^2G^2G_x. \tag{55}$$

From the last equation it follows that for $F = -1$, our equations reduce to the supersymmetric Korteweg–de Vries equation. As is known, there are three different generalizations of the extended supersymmetric KdV equation which have a Lax representation [27–29, 42, 46] and this can be written down compactly as

$$\frac{\mathrm{d}}{\mathrm{d}t}G = \left(-G_{xx} + 3G\,(\mathcal{D}_1\mathcal{D}_2 G) + \tfrac{1}{2}(\alpha-1)\left(\mathcal{D}_1\mathcal{D}_2 G^2\right) + \alpha G^3\right)_x . \tag{56}$$

Here α is just a free parameter which enumerates these three different cases. Our case corresponds to $\alpha = 1$, after rescaling the time and transforming G into $-G$. In [46] the author considered the non-standard Lax representations for this equation. Here, as a byproduct of our analysis we obtained the usual Lax representation for this equation which can be connected with the extended supersymmetric AKNS approach. Indeed, our Lax operator in this case takes the form

$$L = \partial - \partial^{-1}\mathcal{D}_1\mathcal{D}_2 G \tag{57}$$

with the following flow:

$$L_t = \left[\left(L^3\right)_+, L\right] . \tag{58}$$

Unfortunately, similar to the superfermionic case considered in section 4, we have not found the bi-Hamiltonian structure of this equation.

6. The superfermionic and superbosonic approach

We are now able to consider the mixed approach to the construction of the SUSY multicomponent KP hierarchy. Therefore we now consider the following SUSY Lax operator:

$$L = \partial + \sum_{i=1}^{k} F_i\partial^{-1}D_1D_2G_i + \sum_{j=1}^{m} B_j\partial^{-1}D_1D_2C_j \tag{59}$$

where F and G are now vector superfermions with the expansions (21), (22), while B and C are superbosons with the expansions (49), (50). Using the same technique as in the previous sections we computed the second and third flows, but the final formulae are complicated. Hence we only present the second flow, which can be written down as

$$\frac{\mathrm{d}}{\mathrm{d}t}F_i = F_{ixx} - F_iZ + 2\sum_{l=1}^{k} F_l\,(\mathcal{D}_1\mathcal{D}_2 G_lF_i) + 2\sum_{j=1}^{m} B_j\left(\mathcal{D}_1\mathcal{D}_2 C_jF_i\right) \tag{60}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}B_j = B_{jxx} - B_jZ + 2\sum_{l=1}^{k} F_l\left(\mathcal{D}_1\mathcal{D}_2 G_lB_j\right) + 2\sum_{s=1}^{m} B_s\left(\mathcal{D}_1\mathcal{D}_2 C_sB_j\right) \tag{61}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}G_i = -G_{ixx} + G_iZ - 2\sum_{l=1}^{k} G_l\,(\mathcal{D}_1\mathcal{D}_2 G_iF_l) - 2\sum_{j=1}^{m} C_j\left(\mathcal{D}_1\mathcal{D}_2 G_iB_j\right) \tag{62}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}C_j = -C_{jxx} + C_j Z - 2\sum_{l=1}^{k} G_l \left(\mathcal{D}_1\mathcal{D}_2 C_j F_l\right) - 2\sum_{s=1}^{m} C_s \left(\mathcal{D}_1\mathcal{D}_2 C_j B_s\right) \tag{63}$$

where

$$Z = \left(\sum_{l=1}^{k} F_i G_i + \sum_{j=1}^{m} B_j C_j\right)^2 . \tag{64}$$

As we can see, the last system of equations describes a huge class of interacting fields. In some sense, it describes the interaction of the superfermions with the superbosons.

7. Concluding remarks

We have constructed the extended supersymmetric version of the multicomponent KP hierarchy in three different ways. We obtained a new class of integrable equations for which we were able to construct the Lax operator and showed that they are Hamiltonian equations. Moreover, due to the existence of the Lax operator, we obtained an infinite number of conserved currents for our generalizations.

In soliton theory, in order to prove the involution of the conserved currents, we utilize the recursion operator. Magri [53] has shown that such a recursion operator can be constructed if we know the bi-Hamiltonian structure. However, in our case we cannot find such a bi-Hamiltonian structure. It does not mean that our system does not possess the recursion operator, nor that it is not completely integrable. An excellent example of this situation, where we do not know the bi-Hamiltonian structure but do know the recursion operator, is the Burgers equation [55]. Therefore, it seems reasonable that if we wish to prove the commutativity of the conservation laws we should try quite different methods of factorization of the recursion operator by the bi-Hamiltonian operators. Moreover, to prove the commutativity of the conservation laws, it is not necessary to have a bi-Hamiltonian formulation: an argument based on the Lax formalism is also possible [59, 27].

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